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PRINJECTIVE MODULES, REFLECTION FUNCTORS, QUADRATIC FORMS, AND AUSLANDER-REITEN SEQUENCES

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ABSTRACT. Let A, B be artinian rings and let ${}_AM_B$ be an $(A-B)$-bimodule which is a finitely generated left A-module and a finitely generated right B-module. A right ${}_AM_B$-prinjective module is a finitely generated module $X_R = (X'_A, X''_B, \varphi: X'_A \otimes_A M_B \to X''_B)$ over the triangular matrix ring

$$R = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$$

such that X'_A is a projective A-module, X''_B is an injective B-module, and φ is a B-homomorphism.

We study the category $\operatorname{prin}(R)^A_B$ of right ${}_AM_B$-prinjective modules. It is an additive Krull-Schmidt subcategory of $\operatorname{mod}(R)$ closed under extensions. For every X, Y in $\operatorname{prin}(R)^A_B$, $\operatorname{Ext}^2_R(X, Y) = 0$. When R is an Artin algebra, the category $\operatorname{prin}(R)^A_B$ has Auslander-Reiten sequences and they can be computed in terms of reflection functors. In the case that R is an algebra over an algebraically closed field we give conditions for $\operatorname{prin}(R)^A_B$ to be representation-finite or representation-tame in terms of a Tits form. In some cases we calculate the coordinates of the Auslander-Reiten translation of a module using a Coxeter linear transformation.

0. INTRODUCTION

Let A, B be artinian rings and let ${}_AM_B$ be an $(A-B)$-bimodule which is a finitely generated left A-module and a finitely generated right B-module. By a right ${}_AM_B$-prinjective module we shall mean a finitely generated module

$$X_R = (X'_A, X''_B, \varphi: X'_A \otimes_A M_B \to X''_B)$$

over the triangular matrix ring

$$R = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix} \tag{0.1}$$

such that X'_A is a projective A-module, X''_B is an injective B-module, and φ is a B-homomorphism.

The aim of this paper is to study the category $\operatorname{prin}(R)^A_B$ of right ${}_AM_B$-prinjective modules. It is an additive Krull-Schmidt subcategory of $\operatorname{mod}(R)$ closed under extensions. It has enough relative projective and relative injective modules and it is a hereditary subcategory of $\operatorname{mod}(R)$ in the sense that $\operatorname{Ext}^2_R(X, Y) = 0$ for all X and Y in $\operatorname{prin}(R)^A_B$.

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We prove in this paper that if R is an Artin algebra then $\operatorname{prin}(R)_B^A$ has Auslander-Reiten sequences and we show how to compute them in terms of reflection functors. In the case that R is an algebra over an algebraically closed field we give necessary conditions for $\operatorname{prin}(R)_B^A$ to be a representation-finite or a representation-tame category in terms of a Tits form. Moreover, if $\operatorname{prin}(R)_B^A$ has a preprojective component, this gives a criterion for the representation-finiteness of $\operatorname{prin}(R)_B^A$. In the latter case, we calculate the coordinates of the Auslander-Reiten translation of a module by means of a Coxeter linear transformation.

The main motivation of our study is the fact observed in [10, 18, 19] (see also §1) that given finite Krull-Schmidt categories $\mathbb{K}$, $\mathbb{L}$ and a $(\mathbb{K}-\mathbb{L})$-bimodule $N\colon \mathbb{K}^{\mathrm{op}} \times \mathbb{L} \to \operatorname{mod}(k)$ there is an algebra R of the form (0.1) and an equivalence of categories

$$\operatorname{Mat}({}_{\mathbb{K}}N_{\mathbb{L}}) \xrightarrow{\sim} \operatorname{prin}(R)_B^A,$$

where $\operatorname{Mat}({}_{\mathbb{K}}N_{\mathbb{L}})$ is the category of ${}_{\mathbb{K}}N_{\mathbb{L}}$-matrices in the sense of Drozd [7]. In particular, the subspace category $\mathscr{U}(\mathbb{K}_F)$ of a vector space category $\mathbb{K}_F$ is equivalent to $\operatorname{prin}\left(\begin{smallmatrix} E & M \\ 0 & F \end{smallmatrix}\right)$ for some ${}_EM_F$. Therefore, our results give us tools to study the representation type of $\operatorname{Mat}({}_{\mathbb{K}}N_{\mathbb{L}})$, which turns out to be an important class of matrix problems with many useful applications [16, 19].

The notion of an ${}_AM_B$-prinjective module was introduced in [10] and independently in [19] under the name of ${}_AM_B$-matrix module. The category $\operatorname{prin}(R)_B^A$ is denoted by $\operatorname{mod}_{\mathrm{in}}^{\mathrm{pr}}(R)$ in [19]. Our Coxeter scheme, studied in §4, generalizes the one defined in [17] for the case where B is a division ring.

Throughout this paper R is an Artin algebra of the form (0.1). We denote by $\operatorname{mod}(R)$, $\operatorname{pr}(A)$, and $\operatorname{inj}(B)$ the categories of finitely generated right R-modules, projective right A-modules, and injective right B-modules, respectively. We assume that A and B are basic algebras and we fix complete sets $\{e_1, \dots, e_n\}$ and $\{\eta_1, \dots, \eta_m\}$ of orthogonal primitive idempotents in A and B respectively. Given $X_R = (X'_A, X''_B, \varphi)$ in $\operatorname{mod}(R)$ we put $X'_i = X'e_i = Xe_i$ and $X''_j = X''\eta_j = X\eta_j$.

We fix a duality $D\colon \operatorname{mod}(R) \to \operatorname{mod}(R^{\mathrm{op}})$; that is,

$$D(-) = \operatorname{Hom}_C(-, E(C/\mathscr{J}(C))),$$

where C is a commutative artinian ring contained in the center of R such that R is a finitely generated C-module, $\mathscr{J}(C)$ is its Jacobson radical, and $E(-)$ denotes the injective envelope. We denote by

$$\mathfrak{N}\colon \operatorname{pr}(A) \to \operatorname{inj}(A) \tag{0.2}$$

the Nakayama equivalence defined by $\mathfrak{N}(P) = D\operatorname{Hom}_A(P, A)$ (we will write $\mathfrak{N}_A$ whenever confusion may arise). We will frequently use the equivalence

$$P \otimes_A M_B \xrightarrow{\sim} \operatorname{Hom}_A({}_B\widetilde{M}_A, \mathfrak{N}(P)), \tag{0.3}$$

with $\widetilde{M} = DM$ and P in $\operatorname{pr}(A)$, defined as the composed map

$$P \otimes_A M_B \xrightarrow{\sigma} \operatorname{Hom}_A(\operatorname{Hom}_A(P, A), {}_AM_B) \xrightarrow{D} \operatorname{Hom}_A({}_B\widetilde{M}_A, \mathfrak{N}(P)),$$

with $\sigma(p \otimes m)(f) = f(p)m$.

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1. The category of ${}_AM_B$-matrices

Let us recall from [7] the definition of $\mathrm{Mat}({}_{\mathbb{K}}N_{\mathbb{L}})$.

Suppose that k is a commutative artinian ring, $\mathbb{K}$, $\mathbb{L}$ are finite Krull-Schmidt k-categories, and $N: \mathbb{K}^{\mathrm{op}} \times \mathbb{L} \to \mathrm{mod}(k)$ is a $(\mathbb{K} - \mathbb{L})$-bimodule, i.e., it is an additive k-bilinear functor. The objects of $\mathrm{Mat}({}_{\mathbb{K}}N_{\mathbb{L}})$ are triples (x, y, f) with $x \in \mathcal{O}b\mathbb{K}$, $y \in \mathcal{O}b\mathbb{L}$, and $f \in N(x, y)$. A morphism from (x, y, f) to (x', y', f') is a pair (φ, ψ), where $\varphi \in \mathbb{K}(x, x')$, $\psi \in \mathbb{L}(y, y')$ are such that $N(x, \psi)f = N(\varphi, y')f' \in N(x, y')$. It is known that $\mathrm{Mat}({}_{\mathbb{K}}N_{\mathbb{L}})$ is an additive Krull-Schmidt category.

Following [18, 19] we associate to ${}_{\mathbb{K}}N_{\mathbb{L}}$ an Artin k-algebra $R = \left(\begin{smallmatrix} A & {}_AM_B \\ 0 & B \end{smallmatrix}\right)$ as follows. We take complete sets $K_1, \dots, K_n$ and $L_1, \dots, L_m$ of pairwise nonisomorphic indecomposable objects in $\mathbb{K}$ and $\mathbb{L}$ respectively and we set

$$K = K_1 \oplus \cdots \oplus K_n, \qquad L = L_1 \oplus \cdots \oplus L_m,$$
$$A = \mathbb{K}(K, K), \quad B = \mathbb{L}(L, L), \quad \text{and} \quad {}_AM_B = DN(K, L).$$

Proposition 1.1. *With the notation above, there is an equivalence of categories*

$$\mu^*: \mathrm{Mat}({}_{\mathbb{K}}N_{\mathbb{L}}) \xrightarrow{\sim} \mathrm{prin}(R)^A_B.$$

Proof. We repeat the proof given in [19, §5; 10, §1]. Let $\omega: \mathbb{K} \to \mathrm{pr}(A)$ and $\omega': \mathbb{L} \to \mathrm{pr}(B)$ be the Yoneda equivalences given by $\omega(-) = \mathbb{K}(K, -)$ and $\omega'(-) = \mathbb{L}(L, -)$. For $x \in \mathcal{O}b\mathbb{K}$ and $y \in \mathcal{O}b\mathbb{L}$ the Yoneda lemma yields a natural isomorphism

$$\mu: N(x, y) \to \mathrm{Hom}_B(\omega(x) \otimes_A M_B, \mathfrak{N}\omega'(y)) \tag{1.2}$$

which is the composed map

$$\begin{aligned} N(x, y) &\xrightarrow{\sim} \mathrm{Nat}(\mathbb{L}(y, -), N(x, -)) \xrightarrow{\sim} \mathrm{Hom}_B(\mathbb{L}(y, L), N(x, L)) \\ &\xrightarrow{\sim} \mathrm{Hom}_B(DN(x, L), D\mathbb{L}(y, L)) \\ &\xrightarrow{\sim} \mathrm{Hom}_B(\mathbb{K}(-, x) \otimes_{\mathbb{K}} DN(-, L), \mathfrak{N}\omega'(y)) \\ &\xrightarrow{\sim} \mathrm{Hom}_B(\omega(x) \otimes_A M_B, \mathfrak{N}\omega'(y)), \end{aligned}$$

where $\mathrm{Nat}(-, -)$ denotes the set of k-linear natural transformations.

Now we associate the module $\mu^*(x, y, f) = (\omega(x), \mathfrak{N}\omega'(y), \mu(f))$ in $\mathrm{prin}(R)^A_B$ to (x, y, f) in $\mathrm{Mat}({}_{\mathbb{K}}N_{\mathbb{L}})$ with $f \in N(x, y)$. Applying (1.2) one can easily check that μ^* defines an equivalence of categories. □

2. The category of prinjective modules

Let $R = \left(\begin{smallmatrix} A & {}_AM_B \\ 0 & B \end{smallmatrix}\right)$ be an Artin algebra. A module P (resp. Q) in $\mathrm{prin}(R)^A_B$ will be called *prin-projective* (resp. *prin-injective*) if $\mathrm{Hom}_R(P, -)$ (resp. $\mathrm{Hom}_R(-, Q)$) carries over short exact sequences in $\mathrm{mod}(R)$ with prinjective terms into exact ones.

To describe the indecomposable prin-projective modules and prin-injective R-modules we need the following notation. Given $X = (X', X'', \varphi)$ in $\operatorname{mod}(R)$ we form two modules,

(2.1) $$\widehat{X} = (X', E(X''), \hat{\varphi}) \quad \text{and} \quad \widetilde{X} = (P(X'), X'', \tilde{\varphi}),$$

where $E(X'')$ denotes the injective envelope of X'' in $\operatorname{mod}(B)$ and $P(X')$ denotes the projective cover of X' in $\operatorname{mod}(A)$; the maps $\hat{\varphi}$ and $\tilde{\varphi}$ are defined in the obvious way. Now we form two families of indecomposable prinjective R-modules,

(2.2) $$\widehat{P}_1, \dots, \widehat{P}_n, {}^0I_1, \dots, {}^0I_m,$$

(2.3) $${}^0P_1, \dots, {}^0P_n, \widetilde{Q}_1, \dots, \widetilde{Q}_m,$$

where $\widehat{P}_j = \widehat{e_jR}$, ${}^0I_t = (0, E(\operatorname{top}\eta_tB), 0)$, ${}^0P_j = (e_jA, 0, 0)$, and $\widetilde{Q}_t = E_R(\operatorname{top}\eta_tB)^\sim$, $\operatorname{top}\eta_tB = \eta_tB/\operatorname{rad}\eta_tB$; $E_R(-)$ denotes the injective envelope in $\operatorname{mod}(R)$.

Proposition 2.4. (a) *The category* $\operatorname{prin}(R)_B^A$ *is closed under extensions in* $\operatorname{mod}(R)$ *and for any* X *in* $\operatorname{prin}(R)_B^A$ *there are exact sequences in* $\operatorname{mod}(R)$

$$0 \to H_1 \to H_0 \to X \to 0,$$
$$0 \to X \to U_0 \to U_1 \to 0,$$

where H_0 *and* U_0 *are direct sums of modules in* (2.2) *and* (2.3) *respectively, whereras* H_1 *and* U_1 *are direct sums of modules* ${}^0I_1, \dots, {}^0I_m$ *and* ${}^0P_1, \dots, {}^0P_n$ *respectively.*

(b) *A module* P *in* $\operatorname{prin}(R)_B^A$ *is prin-projective if and only if* $\operatorname{Ext}_R^1(P, X) = 0$ *for all* X *in* $\operatorname{prin}(R)_B^A$. *The modules* (2.2) *form a complete list of pairwise nonisomorphic indecomposable prin-projective modules.*

(c) *A module* Q *in* $\operatorname{prin}(R)_B^A$ *is prin-injective if and only if* $\operatorname{Ext}_R^1(X, Q) = 0$ *for all* X *in* $\operatorname{prin}(R)_B^A$. *The modules* (2.3) *form a complete list of pairwise nonisomorphic indecomposable prin-injective modules.*

(d) $\operatorname{prin}(R)_B^A$ *is a hereditary subcategory of* $\operatorname{mod}(R)$, *i.e.,* $\operatorname{Ext}_R^2(X, Y) = 0$ *for all* X *and* Y *in* $\operatorname{prin}(R)_B^A$.

Proof. (a) Let $X = (X', X'', \varphi)$ be in $\operatorname{prin}(R)_B^A$. Since X' is in $\operatorname{pr}(A)$, then $L = (X', X' \otimes_A M_B, \mathrm{id})$ is in $\operatorname{pr}(R)$ and therefore $\widehat{L}$ is a direct sum of copies of $\widehat{P}_1, \dots, \widehat{P}_n$. The maps $\mathrm{id}_{X'}$ and $\mathrm{id}_{X''}$ induce the exact sequence

$$0 \to (0, Y, 0) \to \widehat{L} \oplus (0, X'', 0) \to X \to 0.$$

Since obviously Y, X'' are in $\operatorname{inj}(B)$, we then get the first sequence in (a). The second one arises dually.

(b) It is clear that $\operatorname{Ext}_R^1(P, X) = 0$ for all X in $\operatorname{prin}(R)_B^A$ and P of the form (2.2). Then statement (b) follows from (a).

Since (c) follows similarly from (a) and (d) is a consequence of (a)–(c), the proof is complete. □

Let X be indecomposable. A map $f: X \to Y$ in an additive Krull-Schmidt category K $(= \operatorname{prin}(R)_B^A, \operatorname{mod}(R), \dots)$ is called a *source map* for X if it satisfies:

(a) f is not split mono;
(b) given $f'\colon X \to Y'$ not split mono, there exists $\eta\colon Y \to Y'$ with $f' = \eta f$; and
(c) if $\gamma \in \operatorname{End}_K(Y)$ satisfies $f\gamma = f$, then γ is an automorphism.

The dual concept is a *sink map* (see [16, (2.2)]).

We describe now the sink morphisms in $\operatorname{prin}(R)_B^A$ ending at the prin-projective indecomposable modules.

Consider the case $\widehat{P}_j = \widehat{e_jR} = (e_jA, E(e_jM), \widehat{\mathrm{id}})$. Let $\xi\colon X \to e_jA$ be a sink morphism in $\operatorname{pr}(A)$; that is, $\xi\colon X = P(\operatorname{rad} e_jA) \twoheadrightarrow (\operatorname{rad} e_jA) \hookrightarrow e_jA$. Then

$$(2.5) \qquad (\xi, \mathrm{id})\colon (X, e_jM, \xi \otimes 1_M)^\frown \to \widehat{P}_j$$

is a sink morphism.

Consider the case ${}^0I_t = (0, E(\operatorname{top} \eta_t B), 0)$. Let $\lambda\colon Y \to \mathfrak{N}^{-1}E(\operatorname{top}\eta_t B)$ be a sink morphism in $\operatorname{pr}(B)$. Then $\nu = \mathfrak{N}\lambda\colon J = \mathfrak{N}Y \to E(\operatorname{top}\eta_t B)$ is a sink morphism in $\operatorname{inj}(B)$. Let $j\colon \ker\nu \hookrightarrow J$ be the inclusion. Then

$$(2.6) \qquad (0, \nu)\colon (\operatorname{Hom}_B(M, \ker\nu), J, \overline{j})^\sim \to {}^0I_t,$$

where $\overline{j}$ is the adjoint map to $\operatorname{Hom}_B(M, j)$, is a sink morphism.

The description of the source morphisms in $\operatorname{prin}(R)_B^A$ starting at ${}^0P_j$ and $\widetilde{Q}_t$ is dual.

3. Auslander-Reiten sequences for prinjective modules

Let R be an Artin algebra of the form (0.1).

In this section we establish the existence of Auslander-Reiten sequences in $\operatorname{prin}(R)_B^A$ (even in two different ways!) and we give some useful relations of the category $\operatorname{prin}(R)_B^A$ with other module categories.

We start by recalling some definitions from [19]. By $\operatorname{mod}^{pg}(R)^A$ we denote the full subcategory of $\operatorname{mod}(R)$ consisting of modules of the form $X = (X', X'', \varphi)$ such that $X' \in \operatorname{pr}(A)$ and $\varphi\colon X' \otimes_A M \to X''$ is onto. Dually, $\operatorname{mod}_{ic}(R)_B$ is the full subcategory of $\operatorname{mod}(R)$ consisting of modules $X = (X', X'', \varphi)$ in $\operatorname{mod}(R)$ such that $X'' \in \operatorname{inj}(B)$ and the adjoint map to φ, $\overline{\varphi}\colon X' \to \operatorname{Hom}_B(M, X'')$ is mono. Observe that modules in $\operatorname{mod}^{pg}(R)^A$ have no top at B and modules in $\operatorname{mod}_{ic}(R)_B$ have no socle at A. The category of *adjusted modules* $\operatorname{adj}(R)_B^A$ consists of finitely generated modules of the form $X = (X', X'', \varphi)$ such that φ is onto and $\overline{\varphi}$ is mono.

Let $\Theta_B\colon \operatorname{prin}(R)_B^A \to \operatorname{mod}^{pg}(R)^A$ be the functor given by $(X', X'', \varphi) \mapsto (X', \operatorname{Im}\varphi, \operatorname{res}\varphi)$. Dually, $\Theta^A\colon \operatorname{prin}(R)_B^A \to \operatorname{mod}_{ic}(R)_B$ is given by

$$(X', X'', \varphi) \mapsto (\operatorname{Im}\overline{\varphi}, X'', j_\varphi),$$

where j_φ is the adjoint map to the inclusion $\operatorname{Im}\overline{\varphi} \hookrightarrow \operatorname{Hom}_A(M, X'')$. We get the following commutative diagram:

$$(3.1) \qquad \begin{array}{ccccc} & & \operatorname{mod}^{pg}(R)^A & & \\ & \overset{\Theta_B}{\nearrow} & & \overset{\Theta^A}{\searrow} & \\ \operatorname{prin}(R)_B^A & & & & \operatorname{adj}(R)_B^A \\ & \underset{\Theta^A}{\searrow} & & \underset{\Theta_B}{\nearrow} & \\ & & \operatorname{mod}_{ic}(R)_B & & \end{array}$$

Clearly, we get

(3.2) $\Theta^A(\widetilde{X}) \xrightarrow{\sim} X$, $\Theta_B(\widehat{Y}) \xrightarrow{\sim} Y$ for $X \in \mathrm{mod}_{ic}(R)_B$, $Y \in \mathrm{mod}^{pg}(R)^A$.

Lemma 3.3. (a) *If* $X \in \mathrm{mod}^{pg}(R)^A$ *(or* $X \in \mathrm{prin}(R)^A_B$*), then there exists a natural epimorphism* $\varepsilon_X\colon X \to \Theta^A(X)$ *such that for any morphism* $f\colon Z \to \Theta^A(X)$ *with* $Z \in \mathrm{mod}^{pg}(R)^A$ *(or* $Z \in \mathrm{prin}(R)^A_B$*), there is a lifting* $\tilde{f}\colon Z \to X$ *with* $\varepsilon_X \tilde{f} = f$.

(b) *If* $X \in \mathrm{mod}_{ic}(R)_B$ *(or* $X \in \mathrm{prin}(R)^A_B$*), then there exists a natural monomorphism* $\eta_X\colon \Theta_B(X) \to X$ *such that for any morphism* $f\colon \Theta_B(X) \to Z$ *with* $Z \in \mathrm{mod}_{ic}(R)_B$ *(or* $Z \in \mathrm{prin}(R)^A_B$*), there is an extension* $\hat{f}\colon X \to Z$ *with* $\hat{f}\eta_X = f$.

(c) *If* $e\colon 0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ *is an exact sequence in* $\mathrm{adj}(R)^A_B$ *(resp. in* $\mathrm{mod}_{ic}(R)_B$*), then there exists an exact sequence* $\tilde{e}\colon 0 \to \widetilde{X} \xrightarrow{\mu} E \xrightarrow{\nu} \widetilde{Z} \to 0$ *in* $\mathrm{mod}^{pg}(R)^A$ *(resp. in* $\mathrm{prin}(R)^A_B$*) such that* $\Theta^A(\tilde{e})$ *and* e *are equivalent sequences.*

(d) *If* $e\colon 0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ *is an exact sequence in* $\mathrm{adj}(R)^A_B$ *(resp. in* $\mathrm{mod}^{pg}(R)^A$*), then there exists an exact sequence* $\hat{e}\colon 0 \to \widehat{X} \xrightarrow{\mu} L \xrightarrow{\nu} \widehat{Z} \to 0$ *in* $\mathrm{mod}_{ic}(R)_B$ *(resp. in* $\mathrm{prin}(R)^A_B$*) such that* $\Theta_B(\hat{e})$ *and* e *are equivalent sequences.*

(e) *Let* X *be an indecomposable in* $\mathrm{mod}^{pg}(R)^A$ *(or in* $\mathrm{prin}(R)^A_B$*). Then* $\Theta^A(X) = 0$ *if and only if* $X \simeq {}^0P_j$ *for some* j*. If* $X \not\simeq {}^0P_j$*, then* $\Theta^A(X)$ *is indecomposable and* $\widetilde{\Theta^A(X)} \simeq X$*. Moreover,* $\ker\Theta^A = [{}^0P_1, \dots, {}^0P_n]$*, i.e.,* $\Theta^A(f) = 0$ *if and only if* f *factorizes through a direct sum of* ${}^0P_1, \dots, {}^0P_n$.

(f) *Let* X *be an indecomposable in* $\mathrm{mod}_{ic}(R)_B$ *(or in* $\mathrm{prin}(R)^A_B$*). Then* $\Theta_B(X) = 0$ *if and only if* $X \simeq {}^0I_t$ *for some* t*. If* $X \not\simeq {}^0I_t$*, then* $\Theta_B(X)$ *is indecomposable and* $\widehat{\Theta_B(X)} \simeq X$*. Moreover,* $\ker\Theta_B = [{}^0I_1, \dots, {}^0I_m]$.

(g) *Let* $X \in \mathrm{mod}^{pg}(R)^A$ *(resp.* $X \in \mathrm{prin}(R)^A_B$*) be an indecomposable such that* $X \not\simeq {}^0P_j$ $(1 \le j \le n)$*. Then* $\Theta^A(X)$ *is projective in* $\mathrm{adj}(R)^A_B$ *(resp. in* $\mathrm{mod}_{ic}(R)_B$*) if and only if* X *is projective in* $\mathrm{mod}^{pg}(R)^A$ *(resp. prin-projective).*

(h) *Let* $X \in \mathrm{mod}_{ic}(R)_B$ *(resp.* $X \in \mathrm{prin}(R)^A_B$*) be an indecomposable such that* $X \not\simeq {}^0I_t$ $(1 \le t \le m)$*. Then* $\Theta_B(X)$ *is injective in* $\mathrm{adj}(R)^A_B$ *(resp. in* $\mathrm{mod}^{pg}(R)^A$*) if and only if* X *is injective in* $\mathrm{mod}_{ic}(R)_B$ *(resp. prin-injective).*

(i) *The functors* Θ^A, Θ_B *are full and dense with*

$$\ker\Theta^A\Theta_B = [{}^0P_1, \dots, {}^0P_n, {}^0I_1, \dots, {}^0I_m],$$

and

$$\mathrm{prin}(R)^A_B/[{}^0P_1, \dots, {}^0P_n, {}^0I_1, \dots, {}^0I_m] \simeq \mathrm{adj}(R)^A_B.$$

Proof. We give some indications of the proofs of (a), (c), (e), (g), (i); the other claims are dual.

(a) Let $X = (X', X'', \varphi) \in \mathrm{mod}^{pg}(R)^A$. Then there is an onto A-morphism $\nu\colon X' \to \mathrm{Im}\,\overline{\varphi}$ such that $\overline{\varphi} = i_\varphi \circ \nu$, where i_φ is the inclusion $\mathrm{Im}\,\overline{\varphi} \hookrightarrow \mathrm{Hom}_B(M, X'')$. Then $\varepsilon_X = (\nu, 1_{X''})\colon X \to \Theta^A(X)$ is a natural epimorphism. If $f = (f', f'')\colon Z \to \Theta^A(X)$ is a morphism with $Z \in \mathrm{mod}^{pg}(R)^A$, then there is an A-morphism $\tilde{f}'\colon Z' \to X'$ such that $f' = \nu\tilde{f}'$. Therefore, $\tilde{f} = (\tilde{f}', f'')\colon Z \to X$ is a morphism such that $f = \varepsilon_X \tilde{f}$.

(c) Consider the following exact and commutative diagram of A-modules:

$$\begin{array}{ccccccccc} 0 & \to & P(X') & \xrightarrow{\mu'} & P & \xrightarrow{\nu'} & P(Z') & \to & 0 \\ & & p_{X'}\downarrow & & p\downarrow & & \downarrow p_{Y'} & & \\ 0 & \to & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $P = P(X') \oplus P(Z')$. We get an exact sequence in $\mathrm{mod}^{pg}(R)^A$,

$$\tilde{e}: 0 \to \widetilde{X} \xrightarrow{(\mu', u'')} E = (P, Y'', P \otimes_A M \xrightarrow{p\otimes 1_M} Y' \otimes_A M \xrightarrow{\varphi_Y} Y'') \xrightarrow{(\nu', v'')} \widetilde{Z} \to 0.$$

Since $\overline{\varphi_Y(p \otimes 1_M)} = \overline{\varphi}_Y p$, then $\Theta^A(E) = \Theta^A(\widetilde{Y}) \xrightarrow{\sim} Y$. It follows easily that $\Theta^A(\tilde{e})$ and e are equivalent sequences.

(e) This is left to the reader.

(g) Assume that $X \in \mathrm{mod}^{pg}(R)^A$ is an indecomposable such that $X \not\simeq {}^0P_j$. Then $\Theta^A(X) \neq 0$. Assume that $\Theta^A(X)$ is projective in $\mathrm{adj}(R)^A_B$.

Let $e: 0 \to V \xrightarrow{u} E \xrightarrow{v} X \to 0$ be an exact sequence in $\mathrm{mod}^{pg}(R)^A$. Consider the following induced exact and commutative diagram of A-modules:

$$\begin{array}{ccccccccc} 0 & \to & K & \xrightarrow{j} & \mathrm{Im}\,\overline{\varphi}_E & \to & \mathrm{Im}\,\overline{\varphi}_X & \to & 0 \\ & & i\downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}_B(M, V'') & \to & \mathrm{Hom}_B(M, E'') & \to & \mathrm{Hom}_B(M, X'') & & \end{array}$$

We get an exact sequence in $\mathrm{adj}(R)^A_B$

$$0 \to (K, V'', \psi) \xrightarrow{(j, v'')} \Theta^A(E) \xrightarrow{\Theta^A v} \Theta^A(X) \to 0,$$

where $\overline{\psi} = i$. Therefore, there exists $h: \Theta^A(X) \to \Theta^A(E)$, a morphism such that $\Theta^A(\mu)h = 1_{\Theta^A(X)}$. By (a), there is a lifting $\tilde{h}: X \to E$ such that $\varepsilon_Y \tilde{h} = h\varepsilon_X$. The endomorphism $\mu\tilde{h}$ satisfies $\Theta^A(\mu\tilde{h}) = 1_{\Theta^A(X)}$. Since X is indecomposable, $\nu\tilde{h}$ is an isomorphism. Thus μ splits and X is projective.

(i) Follows easily. □

If $\eta: 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact sequence in $\mathrm{mod}(R)$, where X, Y, Z are in K $(= \mathrm{adj}(R)^A_B, \mathrm{prin}(R)^A_B, \ldots)$ with X and Z indecomposable, then η is an *Auslander-Reiten sequence* if f is a source map, or, equivalently, g is a sink map in K. The category K is said to *have Auslander-Reiten sequences* if for every indecomposable noninjective (resp. nonprojective) object X in K there exists an Auslander-Reiten sequence in K starting (resp. ending) at X (see [16]). Moreover, K is said to *have source maps* (resp. *sink maps*) if for every indecomposable object X in K there exists a source map in K staring (resp. sink map in K ending) at X.

Theorem 3.4. *The categories* $\mathrm{adj}(R)^A_B$, $\mathrm{mod}^{pg}(R)^A$, $\mathrm{mod}_{ic}(R)_B$, *and* $\mathrm{prin}(R)^A_B$ *have Auslander-Reiten sequences, source maps, and sink maps.*

Proof. As observed in [19], the existence of Auslander-Reiten sequences in $\mathrm{adj}(R)^A_B$ is a direct application of [2].

Let X be a nonprojective indecomposable in $\mathrm{mod}_{ic}(R)_B$. By Lemma 3.3, $\Theta_B(X)$ is indecomposable and nonzero. If $\Theta_B(X)$ is nonprojective, then there exists an Auslander-Reiten sequence $e: 0 \to Z \xrightarrow{u} E \xrightarrow{v} \Theta_B(X) \to 0$ in $\operatorname{adj}(R)_B^A$. By (3.3)(d),(f), there is an exact commutative diagram in $\operatorname{mod}_{ic}(R)_B$ as follows:

$$\begin{array}{ccccccccc} e: 0 & \to & Z & \xrightarrow{u} & E & \xrightarrow{v} & \Theta_B(X) & \to & 0 \\ & & \downarrow & & \downarrow t & & \downarrow \eta_X & & \\ \hat{e}: 0 & \to & \hat{Z} & \to & L & \xrightarrow{\mu} & X \simeq \widehat{\Theta_B(X)} & \to & 0 \end{array}$$

Moreover, $\Theta_B(\hat{e})$ and e are equivalent sequences. We claim that $\hat{e}$ is an Auslander-Reiten sequence in $\operatorname{mod}_{ic}(R)_B$.

In fact, let $f: Y \to X$ be a noninvertible map in $\operatorname{mod}_{ic}(R)_B$, where Y is indecomposable. If $\Theta_B(f) = 0$, then f has a factorization through a direct sum of modules ${}^0I_1, \dots, {}^0I_m$ (3.3)(f). Since every ${}^0I_t$ is projective in $\operatorname{mod}_{ic}(R)_B$, then $f = \mu g$ for some $g: Y \to L$. Suppose now that $\Theta_B(f) \neq 0$. By (3.3)(f), $\Theta_B(X)$ and $\Theta_B(Y)$ are indecomposable and $\Theta_B(f)$ is not invertible. Since e is an Auslander-Reiten sequence, there exists $h: \Theta_B(Y) \to E$ such that $\Theta_B(f) = vh$. By (3.3)(b), there exists $g \in \operatorname{Hom}_R(Y, L)$ such that $th = g\eta_Y$. Hence, $(\mu g - f)\eta_Y = \mu th - \eta_X \Theta_B(f) = \eta_X(vh - \Theta_B(f)) = 0$; that is, $\Theta_B(\mu g - f) = 0$. As above, $\mu g - f$ factorizes through μ. Therefore, f factorizes through μ. Hence our claim follows. If $\Theta_B(X)$ is projective, see the note added in proof.

For the proof that for every noninjective module X in $\operatorname{mod}_{ic}(R)_B$ there exists an Auslander-Reiten sequence starting at X, we recall from [19, §2; 18, §7B] the following. There is a reflection duality

$$D^\bullet: \operatorname{mod}_{ic}(R)_B \to \operatorname{mod}_{ic}(R^\bullet)_B, \tag{3.5}$$

where $R^\bullet$ is the algebra opposite to R^∇ defined in (3.6) below and $D^\bullet$ is the composed functor

$$\operatorname{mod}_{ic}(R)_B \xrightarrow{\nabla_-} \operatorname{mod}^{pg}(R^\nabla)^B \xrightarrow{D} \operatorname{mod}_{ic}(R^\bullet)_B$$

with $\nabla_-(X', X'', \varphi) = (\mathfrak{N}_B^{-1}(X''), \operatorname{coker}\overline{\varphi}, \nu_X)$ and $\nu_X: \mathfrak{N}_B^{-1}(X'') \otimes_B \widetilde{M} \xrightarrow{\sim} \operatorname{Hom}_A(M, X'') \to \operatorname{coker}\overline{\varphi}$.

In view of the duality $D^\bullet$, the required property easily follows.

The existence of source and sink maps in $\operatorname{mod}_{ic}(R)_B$ can be shown similarly as in $\operatorname{prin}(R)_B^A$ as discussed in §2.

The proof for $\operatorname{prin}(R)_B^A$ can be done in a similar way as above, whereas for $\operatorname{mod}^{pg}(R)^A$ the proof follows from the case above using the duality D. □

If X is an indecomposable nonprojective module in $\operatorname{prin}(R)_B^A$ we denote by ΔX the starting term of the Auslander-Reiten sequence ending at X. Dually, if X is indecomposable noninjective, $\Delta^- X$ is the ending term of the Auslander-Reiten sequence in $\operatorname{prin}(R)_B^A$ starting at X.

We introduce a *Coxeter scheme* similar to that given in [17]. For this purpose, we consider the reflection form

$$R^\nabla = \begin{bmatrix} B & {}_B\widetilde{M}_A \\ 0 & A \end{bmatrix} \tag{3.6}$$

of R, where $\widetilde{M} = DM$.

We introduce *reflection functors* in the following way:

$$S^B: \operatorname{prin}(R)_B^A \to \operatorname{mod}_{ic}(R^\nabla)_A$$

defined on objects as

$$X = (X', X'', \varphi) \mapsto (\ker\varphi, \mathfrak{N}_A(X'), j_X),$$

where $j_X\colon \ker\varphi \otimes_B \widetilde{M} \to \mathfrak{N}_A(X')$ is the adjoint map to the composition $\ker\varphi \hookrightarrow X' \otimes_A M \simeq \mathrm{Hom}_A(\widetilde{M}, \mathfrak{N}_A(X'))$;

$$S_A\colon \operatorname{prin}(R)^A_B \to \operatorname{mod}^{pg}(R^\nabla)^B$$

defined on objects as

$$X = (X', X'', \varphi) \mapsto (\mathfrak{N}_B^{-1}(X''), \operatorname{coker}\overline{\varphi}, \nu_X),$$

where $\overline{\varphi}\colon X' \to \mathrm{Hom}_A(M, X'')$ is the adjoint map to φ and ν_X is the composition $\mathfrak{N}_B^{-1}(X'') \otimes_B \widetilde{M} \simeq \mathrm{Hom}_A(M, X'') \twoheadrightarrow \operatorname{coker}\overline{\varphi}$.

We define dually the functors

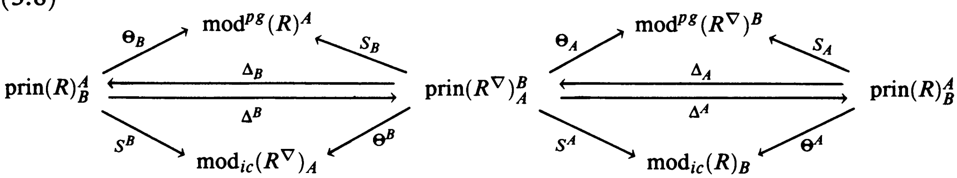
$$S_B\colon \operatorname{prin}(R^\nabla)^B_A \to \operatorname{mod}^{pg}(R)^A, \qquad S^A\colon \operatorname{prin}(R^\nabla)^B_A \to \operatorname{mod}_{ic}(R)_B.$$

The *partial Coxeter maps* are given by

$$(3.7)\qquad \begin{array}{ll} \Delta^B\colon \operatorname{prin}(R)^A_B \to \operatorname{prin}(R^\nabla)^B_A, & \Delta^B(X) = \widetilde{S^B(X)}, \\ \Delta_B\colon \operatorname{prin}(R^\nabla)^B_A \to \operatorname{prin}(R)^A_B, & \Delta_B(Y) = \widehat{S_B(Y)}, \\ \Delta^A\colon \operatorname{prin}(R^\nabla)^B_A \to \operatorname{prin}(R)^A_B, & \Delta^A(Y) = \widetilde{S^A(Y)}, \\ \Delta_A\colon \operatorname{prin}(R)^A_B \to \operatorname{prin}(R^\nabla)^B_A, & \Delta_A(X) = \widehat{S_A(X)}; \end{array}$$

therefore, by (3.2), we get the following commutative diagrams:

(3.8)



We will use this scheme to give formulas connecting the left-hand and right-hand terms of the Auslander-Reiten sequences in $\operatorname{prin}(R)^A_B$. We follow ideas in [19, Proposition 2.17, Theorem 3.28; 17, Corollary 3.7].

By $\operatorname{tr}\colon \operatorname{mod}(R) \to \operatorname{mod}(R^{\mathrm{op}})$ we denote the usual transpose construction.

Lemma 3.9. *Let X be an indecomposable in $\operatorname{prin}(R)^A_B$. Then*

$$D\operatorname{tr}\Theta_B(X) \simeq S^A\Delta^B(X), \qquad \operatorname{tr} D\Theta^A(X) \simeq S_B\Delta_A(X).$$

In particular, $D\operatorname{tr}\Theta_B(X)$ is in $\operatorname{mod}_{ic}(R)_B$ and $\operatorname{tr} D\Theta^A(X)$ is in $\operatorname{mod}^{pg}(R)^A$.

Proof. First we note that the following diagram

$$(3.10)\qquad \begin{array}{ccc} \operatorname{prin}(R^\nabla)^B_A & \xrightarrow{S^A} & \operatorname{mod}_{ic}(R)_B \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ \operatorname{prin}(R^{\nabla\mathrm{op}})^{A^{\mathrm{op}}}_{B^{\mathrm{op}}} & \xrightarrow{S_{A^{\mathrm{op}}}} & \operatorname{mod}^{pg}(R^{\mathrm{op}})^{B^{\mathrm{op}}} \end{array}$$

is commutative. This follows from the definitions of S^A and $S_{A^{\mathrm{op}}}$ and the fact that given $Y_{R^\nabla} = (P_B, Q_A, t)$ in $\operatorname{prin}(R^\nabla)^B_A$ we have

$$D(Y) = (D(Q_A), D(P_B), \psi),$$

where ψ is the image of $t\colon P \otimes_B \widetilde{M}_A \to Q$ under the composed isomorphism

$$(3.11)\qquad \begin{aligned} \operatorname{Hom}_A(P \otimes_B \widetilde{M}_A, Q_A) &\simeq \operatorname{Hom}_A(D(Q_A), D(P \otimes_B \widetilde{M}_A)) \\ &\simeq \operatorname{Hom}_A(D(Q_A), \operatorname{Hom}_B(P_B, {}_AM_B)) \\ &\simeq \operatorname{Hom}_A(D(Q_A), \operatorname{Hom}_B({}_B\widetilde{M}_A, D(P_B))) \\ &\simeq \operatorname{Hom}_B({}_B\widetilde{M}_A \otimes_A D(Q_A), D(P_B)). \end{aligned}$$

Now, let $X_R = (X'_A, X''_B, \varphi)$ be in $\operatorname{prin}(R)^A_B$ and let P_B be the projective cover of $\ker\varphi$. We get an exact sequence $P_B \xrightarrow{u} X' \otimes_A M_B \xrightarrow{\varphi} X''_B$ and

$$\Delta^B(X) = (P_B, \mathfrak{N}_A(X'_A), t),$$

where t is the adjoint map to the composition

$$P_B \xrightarrow{u} X' \otimes_A M_B \xrightarrow{\sim} \operatorname{Hom}_A({}_B\widetilde{M}_A, \mathfrak{N}_A(X')).$$

Let us consider the module $Z = S_{A^{\mathrm{op}}} D\Delta^B(X)$ in $\operatorname{mod}^{pg}(R^{\mathrm{op}})^{B^{\mathrm{op}}}$. Note that $Z = ({}_BZ', {}_AZ'', h)$, where ${}_BZ' = \operatorname{Hom}_B(P_B, B)$ and $h\colon {}_AM \otimes_B Z' \to {}_AZ''$ is the cokernel of the composed map

$$\varepsilon\colon \operatorname{Hom}_A(X'_A, A) \xrightarrow{\sim} D\mathfrak{N}_A(X'_A) \xrightarrow{\overline{\psi}} \operatorname{Hom}_B({}_B\widetilde{M}_A, D(P_B)) \overset{(0.3)}{\xrightarrow{\sim}} {}_AM \otimes_B Z',$$

where $\overline{\psi}$ is the adjoint map to the image ψ of t under the isomorphism (3.11).

On the other hand, $P_R(\Theta_B(X)) \simeq (X'_A, X' \otimes_A M_B, \mathrm{id})$ and therefore the sequence

$$(0, P_B, 0) \xrightarrow{(0,u)} P_R(\Theta_B(X)) \to \Theta_B(X) \to 0$$

is exact. It is not hard to show the commutativity of the following diagram:

$$\begin{array}{ccc} \operatorname{Hom}_R(P_R(\Theta_B(X)), R) & \xrightarrow{(0,u)^*} & \operatorname{Hom}_R((0, P_B, 0), R) \\ \downarrow & & \downarrow \\ (0, \operatorname{Hom}_A(X'_A, A), 0) & \xrightarrow{(0,\varepsilon)} & (\operatorname{Hom}_B(P_B, B), {}_AM \otimes_B \operatorname{Hom}_B(P_B, B), \mathrm{id}) \end{array}$$

Therefore, $\operatorname{tr}\Theta_B(X) \simeq \operatorname{coker}(0, u)^* \simeq \operatorname{coker}(0, \varepsilon) \simeq Z$. Since the diagram (3.10) is commutative, we get $S^A\Delta^B(X) \simeq D(Z) \simeq D\operatorname{tr}\Theta_B(X)$, as desired.

The remaining part of the lemma follows in a similar way. □

Lemma 3.12. *Let* $0 \to X \xrightarrow{u} Y \xrightarrow{t} Z \to 0$ *be an Auslander-Reiten sequence in* $\operatorname{mod}(R)$ *with* X, Z *indecomposable.*

(a) *If* X *is in* $\operatorname{mod}_{ic}(R)_B$, *then* Z *is in* $\operatorname{mod}^{pg}(R)^A$. *If* Z *is not of the form* ${}^0P_j$, *then the induced sequence* $0 \to X \to \Theta^A(Y) \to \Theta^A(Z) \to 0$ *is exact.*

(b) *If* Z *is in* $\operatorname{mod}^{pg}(R)^A$, *then* X *is in* $\operatorname{mod}_{ic}(R)_B$. *If* X *is not of the form* ${}^0I_t$, *then the induced sequence* $0 \to \Theta_B(X) \to \Theta_B(Y) \to Z \to 0$ *is exact.*

Proof. (a) If X is in $\operatorname{mod}_{ic}(R)_B$ then $X \simeq \Theta^A(\widetilde{X})$ and by (3.9), $Z \simeq \operatorname{tr} D(X) \simeq \operatorname{tr} D\Theta^A(\widetilde{X}) \simeq S_B\Delta_A(\widetilde{X})$ is in $\operatorname{mod}^{pg}(R)^A$.

Let $X = (X', X'', \varphi)$, $Y = (Y', Y'', \psi)$, and $Z = (Z', Z'', \lambda)$. We have a commutative diagram

$$\begin{array}{ccccccccc} & & & & \ker\overline{\psi} & \xrightarrow{t'_0} & \ker\overline{\lambda} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & X' & \xrightarrow{u'} & Y' & \xrightarrow{t'} & Z' & \to & 0 \\ & & \wr\downarrow\overline{\varphi} & & \overline{\psi}\downarrow & & \downarrow\overline{\lambda} & & \\ & & \operatorname{Im}\overline{\varphi} & \xrightarrow{\overline{u}'} & \operatorname{Im}\overline{\psi} & \xrightarrow{\overline{t}'} & \operatorname{Im}\overline{\lambda} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{Hom}_B(M, X'') & \to & \operatorname{Hom}_B(M, Y'') & \to & \operatorname{Hom}_B(M, Z'') & \to & 0 \end{array}$$

with the second and the last row exact, where $\overline{\varphi}$, $\overline{\psi}$, $\overline{\lambda}$ are the adjoint maps to φ, ψ, λ and $\tilde{u}'$, $\tilde{t}'$, t_0' are maps induced by u, t. It is clear that t_0' and $\tilde{u}'$ are injective, $\tilde{t}'$ is surjective, and $\tilde{t}'\tilde{u}' = 0$. We claim that t_0' is surjective. If $\ker\overline{\lambda} = 0$, this is clear. If $\ker\overline{\lambda} \neq 0$, the inclusion $(\ker\overline{\lambda}, 0, 0) \hookrightarrow Z$ is nonsurjective (since otherwise $Z \simeq {}^0P_j$ for some j) and by the right almost split property, it has a factorization through t. It follows that t_0' is surjective and hence it is an isomorphism. The snake lemma implies that $\ker\tilde{t}' = \operatorname{Im}\tilde{u}'$ and (a) follows. Statement (b) is dual to (a). □

Theorem 3.13. *If* $0 \to X \to Y \to Z \to 0$ *is an Auslander-Reiten sequence in* $\operatorname{prin}(R)_B^A$*, then*

$$X \simeq D\operatorname{tr}\Theta_B(Z)^{\sim} \simeq \Delta^A\Delta^B(Z) \quad \textit{and} \quad Z \simeq \operatorname{tr}D\Theta^A(X)^{\wedge} \simeq \Delta_B\Delta_A(X).$$

Proof. Let X be an indecomposable non-prin-injective module in $\operatorname{prin}(R)_B^A$. By (3.3)(a),(e), $0 \neq \Theta^A(X)$ is indecomposable and not injective in $\operatorname{mod}(R)$ (compare [19, Theorem 3.28]). Let

$$e\colon 0 \to \Theta^A(X) \to W \xrightarrow{v} L \to 0$$

be an Auslander-Reiten sequence in $\operatorname{mod}(R)$.

Assume first that $L \simeq {}^0P_j$. Then as in the proof of (3.4) we get an Auslander-Reiten sequence

$$\tilde{e}\colon 0 \to \widetilde{\Theta^A(X)} \to E \xrightarrow{v} \tilde{L} \to 0.$$

Since $L = \tilde{L}$ and $X \simeq \widetilde{\Theta^A(X)}$, it follows that

$$Z \simeq L = \widehat{L} = \operatorname{tr}D\Theta^A(X)^{\wedge} \underset{(3.9)}{\simeq} S_B\Delta_A(X)^{\wedge} \simeq \Delta_B\Delta_A(X).$$

Assume L is not of the form ${}^0P_j$. By (3.12), we get an exact sequence

$$0 \to \Theta^A(X) \xrightarrow{u} \Theta^A(W) \to \Theta^A(L) \to 0$$

and also an exact nonsplit sequence in $\operatorname{mod}_{ic}(R)_B$,

$$e'\colon 0 \to \Theta^A(X) \xrightarrow{u'} \widehat{\Theta^A(W)} \to \widehat{\Theta^A(L)} \to 0,$$

where u' is the composition of u with the natural embedding $\Theta^A(W) \hookrightarrow \widehat{\Theta^A(W)}$.

Since L is in $\operatorname{mod}^{pg}(R)^A$ by (3.9), $\Theta^A(L)$ is an indecomposable module in $\operatorname{adj}(R)_B^A$ and $\widehat{\Theta^A(L)}$ is indecomposable in $\operatorname{mod}_{ic}(R)_B$. It is easy to check that u' is a source map in $\operatorname{mod}_{ic}(R)_B$ and, therefore, e' is an Auslander-Reiten sequence in $\operatorname{mod}_{ic}(R)_B$. Applying (3.3) (as in the proof of (3.4)), we get an Auslander-Reiten sequence

$$0 \to \widetilde{\Theta^A(X)} \to E \to \widetilde{\widehat{\Theta^A(L)}} \to 0$$

in $\operatorname{prin}(R)_B^A$. Since $\widetilde{\Theta^A(X)} \simeq X$, it follows that

$$Z \simeq \widetilde{\widehat{\Theta^A(L)}} \simeq \widehat{L} \simeq \operatorname{tr}D\Theta^A(X)^{\wedge} \simeq \Delta_B\Delta_A(X).$$

The remaining part is dual. □

We remark that Theorem 3.13 gives an explicit construction of the Auslander-Reiten sequences in $\operatorname{prin}(R)_B^A$ starting with those in $\operatorname{mod}(R)$.

Corollary 3.14. $\Delta = \Delta^A \Delta^B$ *and* $\Delta^- = \Delta_B \Delta_A$. □

We denote by $\underline{\underline{\mathrm{prin}}}(R)_B^A$ (resp. $\overline{\overline{\mathrm{prin}}}(R)_B^A$) the factor category of $\mathrm{prin}(R)_B^A$ modulo the ideal formed by all maps which admit a factorization through a prin-projective (resp. prin-injective) module. The corresponding Hom functor is denoted by $\underline{\underline{\mathrm{Hom}}}$ and $\overline{\overline{\mathrm{Hom}}}$, respectively. We shall prove the relative Auslander-Reiten formula.

Proposition 3.15. (a) *For* $X, Y \in \mathrm{prin}(R)_B^A$ *there are natural isomorphisms*

$$D\,\mathrm{Ext}_R^1(X, Y) \simeq \overline{\overline{\mathrm{Hom}}}_R(Y, \Delta X) \simeq \underline{\underline{\mathrm{Hom}}}_R(\Delta^- Y, X).$$

(b) Δ *and* Δ^- *induce inverse equivalences*

$$\underline{\underline{\mathrm{prin}}}(R)_B^A \underset{\overline{\Delta}^-}{\overset{\overline{\Delta}}{\rightleftarrows}} \overline{\overline{\mathrm{prin}}}(R)_B^A,$$

$$\mathrm{adj}(R)_B^A/[e_1R, \dots, e_nR] \simeq \mathrm{adj}(R)_B^A/[Q_1, \dots, Q_m],$$

where $Q_t = E_R(\mathrm{top}\, \eta_t B)$.

Proof. (a) Let $e\colon 0 \to \Delta X \xrightarrow{u} E \xrightarrow{v} X \to 0$ be an Auslander-Reiten sequence in $\mathrm{prin}(R)_B^A$. Choose $\varphi \in D\,\mathrm{Ext}_R^1(X, \Delta X)$ such that $\varphi(e) \neq 0$. We define a natural morphism

$$h\colon \mathrm{Hom}_R(-, \Delta X) \to D\,\mathrm{Ext}_R^1(X, -)$$

by $h_{\Delta X}(1_{\Delta X}) = \varphi$.

Let $Y \in \mathrm{prin}(R)_B^A$. We show that h_Y is epi by proving that Dh_Y is mono. Let $0 \neq \eta \in \mathrm{Ext}_R^1(X, Y)$. Then we obtain the following exact commutative diagram:

$$\begin{array}{ccccccccc} \eta\colon 0 & \to & Y & \to & E' & \to & X & \to & 0 \\ & & f\downarrow & & \downarrow & & \| & & \\ e\colon 0 & \to & \Delta X & \to & E & \to & X & \to & 0 \end{array}$$

Thus, $f\eta = e$ and $Dh_Y(\eta)(f) = \varphi(f\eta) = \varphi(e) \neq 0$.

Clearly, h factorizes through $\overline{\overline{\mathrm{Hom}}}_R(-, \Delta X)$. Then we get a natural epimorphism $\overline{h}\colon \overline{\overline{\mathrm{Hom}}}_R(-, \Delta X) \to D\,\mathrm{Ext}_R^1(X, -)$.

Let $f \in \mathrm{Hom}_R(Y, \Delta X)$ and assume that f does not admit a factorization through prin-injective modules. By (2.4), there is an exact sequence $\eta\colon 0 \to Y \to U_0 \to U_1 \to 0$, where U_0 and U_1 are prin-injective modules. Therefore, $0 \neq f\eta \in \mathrm{Ext}_R^1(U_1, \Delta X)$ and we get a morphism $g \in \mathrm{Hom}_R(X, U_1)$ such that $(f\eta)g = e$. Thus we have constructed $\eta g \in \mathrm{Ext}_R^1(X, Y)$ such that $h_Y(f)(\eta g) = \varphi(f(\eta g)) = \varphi(e) \neq 0$. This shows that $\overline{h}$ is a natural isomorphism.

The remaining part of the proof is simple. □

In [10], the existence of Auslander-Reiten sequences in $\mathrm{prin}(R)_B^A$ is shown by means of functorial methods. We give here a brief survey of the main ideas.

Let X be any module in $\mathrm{mod}(R)$. We denote the restriction

$$\mathrm{Hom}_R(-, X) | \mathrm{prin}(R)_B^A$$

by $\mathrm{prin}_R(-, X)$.

Proposition 3.16. *Let* $X \in \operatorname{mod}(R)$. *Then there exists a module* $\overline{X}$ *in* $\operatorname{prin}(R)^A_B$ *such that* $\operatorname{prin}_R(-, \overline{X}) \to \operatorname{prin}_R(-, X)$ *is a projective cover in the category of contravariant functors from* $\operatorname{prin}(R)^A_B$ *to* Ab.

Proof. Let

$$X = (X'_A, X''_B, \varphi) \in \operatorname{prin}(R)^A_B.$$

First we show that $\operatorname{Hom}_B(-, X'')|\operatorname{inj}(B)$ is finitely generated in the category of contravariant functors from $\operatorname{inj}(B)$ to Ab. Indeed, let $0 \to X'' \xrightarrow{m} I_0 \xrightarrow{i} I_1$ be an injective presentation of X'' in $\operatorname{mod}(B)$. Consider the exact sequence

$$0 \to K \xrightarrow{j} \mathfrak{N}_B^{-1}(I_0) \xrightarrow{\mathfrak{N}_B^{-1}(i)} \mathfrak{N}_B^{-1}(I_1)$$

and let $P \xrightarrow{p} K$ be a projective cover. As $i\mathfrak{N}_B(j) = \mathfrak{N}_B(\mathfrak{N}_B^{-1}(i)j) = 0$, we get the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathfrak{N}_B(P) & & \\
 & & \downarrow{\scriptstyle \mathfrak{N}_B(p)} & & \\
 & & \mathfrak{N}_B(K) & & \\
 & & \downarrow{\scriptstyle s} & \searrow{\scriptstyle \mathfrak{N}_B(j)} & \\
0 & \longrightarrow & X'' & \xrightarrow{m} & I_0
\end{array}$$

Therefore, $\nu = s\mathfrak{N}_B(p) \colon I = \mathfrak{N}_B(P) \to X''$ yields the wanted morphism.

Consider the fibered product in $\operatorname{mod}(A)$:

$$\begin{array}{ccc}
L & \xrightarrow{h} & \operatorname{Hom}_B(M, I) \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_B(M, \nu)} \\
X' & \xrightarrow{\overline{\varphi}} & \operatorname{Hom}_B(M, X'')
\end{array}$$

where $\overline{\varphi}$ is the adjoint map to φ. Let $P(L) \xrightarrow{l} L$ be a projective cover in $\operatorname{mod}(A)$ and let $\psi \colon P(L) \otimes_A M \to I$ be the adjoint map to $hl \colon P(L) \to \operatorname{Hom}_B(M, I)$. Then

$$f = (tl, \nu) \colon \overline{X} = (P(L), I, \psi) \to X$$

yields an onto transformation

$$(-, f) \colon \operatorname{prin}_R(-, \overline{X}) \to \operatorname{prin}_R(-, X)$$

in the category $(\operatorname{prin}(R)^A_B, Ab)$. The existence of the projective cover easily follows. □

The existence of Auslander-Reiten sequences can be proved as follows. By (3.16), the category of all finitely presented modules $F \colon (\operatorname{prin}(R)^A_B)^{\mathrm{op}} \to Ab$ is abelian. Let $F \colon (\operatorname{prin}(R)^A_B)^{\mathrm{op}} \to Ab$ be a finitely presented functor; then $DF \colon \operatorname{prin}(R)^A_B \to Ab$ is also finitely presented. If $0 \neq F$, a simple quotient of DF provides a simple subfunctor of F. Let Z be an indecomposable non-prinprojective module in $\operatorname{prin}(R)^A_B$. Then the functor $\operatorname{Ext}^1_R(Z, -) \colon \operatorname{prin}(R)^A_B \to Ab$ is finitely presented and nonzero. Let S_X be a simple subfunctor of $\operatorname{Ext}^1_R(Z, -)$

with X an indecomposable module in $\operatorname{prin}(R)_B^A$ such that $S_X(X) \neq 0$. An exact sequence $\varepsilon\colon 0 \to X \to Y \to Z \to 0$ which generates $\operatorname{Im}(S_X \hookrightarrow \operatorname{Ext}_R^1(Z, -))$ is the wanted sequence.

In [10] the following formula is shown by applying the functorial arguments above.

Proposition 3.17. *Let $Z \in \operatorname{prin}(R)_B^A$ be an indecomposable non-prin-projective module. There exists a prin-injective module I such that $\overline{D\operatorname{tr} Z} \xrightarrow{\sim} \Delta Z \oplus I$.* □

The Auslander-Reiten quiver $\Gamma(\operatorname{prin}(R)_B^A)$ of the category $\operatorname{prin}(R)_B^A$ is defined as usual.

4. Algebras over fields

Let R be a basic finite-dimensional algebra over a field k. In the case R is schurian of upper triangular form we will study the representation type of $\operatorname{prin}(R)_B^A$ in terms of a Cartan matrix and a corresponding quadractic form. The methods will be particularly successful when R is of *finite prinjective type* (i.e., there are only finitely many indecomposable modules in $\operatorname{prin}(R)_B^A$, up to isomorphism) and the Auslander-Reiten quiver $\Gamma(\operatorname{prin}(R)_B^A)$ is a preprojective component.

We recall that we have fixed complete sets $e_1, \dots, e_n$ and $\eta_1, \dots, \eta_m$ of pairwise orthogonal primitive idempotents for A and B respectively. We define numbers

$$a_{ij} = \dim_k e_i A e_j, \quad b_{st} = \dim_k \eta_s B \eta_t, \quad c_{is} = \dim_k e_i M \eta_s,$$

for all $i, j \in \{1, \dots, n\}$ and $s, t \in \{1, \dots, m\}$. We consider the bilinear form

$$\langle \cdot, - \rangle_R\colon \mathbb{Z}^{n+m} \times \mathbb{Z}^{n+m} \to \mathbb{Z} \tag{4.1}$$

defined by the formula

$$\langle z, w \rangle_R = \sum_{i,j=1}^{n} a_{ij} z_i w_j - \sum_{i=1}^{n} \sum_{s=1}^{m} c_{is} z_i w_{n+s} + \sum_{s,t=1}^{m} b_{st} z_{n+s} w_{n+t}.$$

The associated quadratic form $\chi_R(z) = \langle z, z \rangle_R$ is called the *Tits form* of $\operatorname{prin}(R)_B^A$.

Given $X = (X', X'', \varphi)$ in $\operatorname{prin}(R)_B^A$, we consider decompositions

$$X' = \bigoplus_{i=1}^{n} (e_i A)^{x_i} \quad \text{and} \quad X'' = \bigoplus_{s=1}^{m} (DB\eta_s)^{x_{n+s}}.$$

Following [4, 17], we define the *coordinate vector* of X as $\mathbf{cdn}\ X = (x_t)_t \in \mathbb{N}^{n+m}$. The *dimension vector* of X is $\mathbf{dim}\ X = ((\dim_k X' e_i)_i, (\dim_k X'' \eta_s)_s) \in \mathbb{N}^{n+m}$.

In the case k is algebraically closed, we say that R is of *tame prinjective type* if for each vector $w \in \mathbb{N}^{n+m}$, there is a finite family $M_1^{(w)}, \dots, M_{s_w}^{(w)}$ of $(k[x] - R)$-bimodules which are free as $k[x]$-modules and such that every indecomposable $X \in \operatorname{prin}(R)_B^A$ with $\mathbf{cdn}\ X = w$ is isomorphic to $N \otimes_{k[x]} M_i^{(w)}$ for some i and some simple $k[x]$-module N.

One of the motivations for considering the Tits form is the following

Proposition 4.2. (a) *If R is of finite prinjective type, then χ_R is weakly positive (i.e., $\chi_R(z) > 0$, for $0 \neq z \in \mathbb{N}^{n+m}$).*

(b) *If k is algebraically closed and R is of tame prinjective type, then χ_R is weakly nonnegative (i.e., $\chi_R(z) \geq 0$, for $z \in \mathbb{N}^{n+m}$).*

Proof. We recall here briefly the arguments given in [5] for (a) and in [15] for (b).

(a) Let $z \in \mathbb{N}^{n+m}$. By V_z we denote the variety of all $X \in \operatorname{prin}(R)_B^A$ with **cdn** $X = z$. Therefore, $\dim V_z = \sum_{i=1}^{n} \sum_{s=1}^{m} c_{is} z_i z_{n+s}$.

Let $P_z = \bigoplus_{i=1}^{n} (e_i A)^{z_i}$ and $I_z = \bigoplus_{s=1}^{m} (DB\eta_s)^{z_{n+s}}$. Isomorphy in V_z is given by the action of the group $G(z)$ with elements of the form $(\alpha \otimes_A \mathrm{id}_M, \beta)$, where $(\alpha, \beta) \in \operatorname{Aut}_A P_z \times \operatorname{Aut}_B I_z$. Hence,

$$\dim G(z) \leq \sum_{i,j=1}^{n} a_{ij} z_i z_j + \sum_{s,t=1}^{m} b_{st} z_{n+s} z_{n+t}.$$

As $k^* \subset G(z)$ acts trivially and there are only finitely many orbits under the action of $G(z)$ on V_z, $\dim G(z) - 1 \geq \dim V_z$. Thus, $\chi_R(z) \geq 1$.

(b) Let $w \in \mathbb{N}^{n+m}$. There are modules $M_1^{(w)}, \ldots, M_{s_w}^{(w)}$ as in the definition. Let $V_1^{(w)}, \ldots, V_{s_w}^{(w)}$ be the varieties of dimension at most 1 in V_w obtained as the images of the functors $M_i^{(w)} \otimes_{k[x]} -$ evaluated in the $k[x]$-simples. For any family of vectors $(w_i)_i$ with $w_i \in \mathbb{N}^{n+m}$, $\sum_i w_i = z$, and numbers $(j_i)_i$ with $j_i \in \{1, \ldots, s_{w_i}\}$, we consider the (algebraic) map $\prod_i V_{j_i}^{(w_i)} \times G(z) \to V_z$, $((M_i)_i, g) \mapsto (\bigoplus_i M_i)^g$. Clearly, any module in V_z is in the image of one of these maps. Since there are only finitely many of these maps and their images are constructible,

$$\dim V_z \leq \max\left\{\sum_i \dim V_{j_i}^{(w_i)}\right\} + \dim G(z).$$

Therefore,

$$\dim V_z \leq |z| + \dim G(z),$$

where $|z| = \sum_{i=1}^{n} z_i + \sum_{s=1}^{m} z_{n+s}$. Thus, $\chi_R(z) \geq -|z|$. If $\chi_R(z) = -s$, for some $s > 0$, then $-l|z| \leq \chi_R(lz) = l^2 \chi_R(z) = -l^2 s$ and $|z| \geq ls$, for every $l \in \mathbb{N}$. A contradiction proving that $\chi_R(z) \geq 0$. □

The following remark in [16, 2.5] is useful: Let $e\colon 0 \to X \to Y \to Z \to 0$ be an exact sequence in $\operatorname{prin}(R)_B^A$. Then there exists a morphism $\delta \in \operatorname{Hom}_A(Z', \operatorname{Hom}_B(M, X''))$ such that e is equivalent to the sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \xrightarrow{\binom{1}{0}} & X' \oplus Z' & \xrightarrow{(0,1)} & Z' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\varphi}_X} & & \downarrow{\scriptstyle \begin{pmatrix} \overline{\varphi}_X & \delta \\ 0 & \overline{\varphi}_Z \end{pmatrix}} & & \downarrow{\scriptstyle \overline{\varphi}_Z} & & \\
0 & \longrightarrow & \operatorname{Hom}_B(M, X'') & \xrightarrow{\binom{1}{0}} & \operatorname{Hom}_B(M, X'') \oplus \operatorname{Hom}_B(M, Z'') & \xrightarrow{(0,1)} & \operatorname{Hom}_B(M, Z'') & \longrightarrow & 0
\end{array}$$

where $X = (X', X'', \varphi_X)$ and $\overline{\varphi}_X$ is the adjoint map to φ_X; similarly for Y, Z.

We write $e = [\delta]$. We get an exact sequence

$$\text{(4.3)} \qquad \begin{aligned} 0 \to \operatorname{Hom}_R(Z, X) &\xrightarrow{\nu_1} \operatorname{Hom}_A(Z', X') \times \operatorname{Hom}_B(Z'', X'') \\ &\xrightarrow{\nu_2} \operatorname{Hom}_A(Z', \operatorname{Hom}_B(M, X'')) \xrightarrow{\nu} \operatorname{Ext}_R^1(Z, X) \to 0 \end{aligned}$$

with $\nu_1(g) = (g', g'')$, $\nu_2(\alpha, \beta) = \overline{\varphi}_X\alpha - \mathrm{Hom}_B(M, \beta)\overline{\varphi}_Z$, and $\nu(\delta) = [\delta]$.

As a direct consequence we have

Proposition 4.4. *Let* $X, Z \in \mathrm{prin}(R)^A_B$. *Then*

$$\langle \mathbf{cdn}\, Z, \mathbf{cdn}\, X\rangle_R = \dim_k \mathrm{Hom}_R(Z, X) - \dim_k \mathrm{Ext}^1_R(Z, X). \quad \square$$

We now develop some general arguments about reflections that will be useful for our considerations of algebras over fields.

Following [14, 20] by a *bipartite Cartan matrix* we shall mean a matrix C of the form

$$(4.5) \qquad C = \left[\begin{array}{cccc|cccc} 1 & a_{12} & \dots & a_{1n} & c_{11} & c_{12} & \dots & c_{1m} \\ a'_{12} & 1 & \dots & a_{2n} & c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a'_{1n} & a'_{2n} & \dots & 1 & c_{n1} & c_{n2} & \dots & c_{nm} \\ \hline c'_{11} & c'_{21} & \dots & c'_{n1} & 1 & b_{12} & \dots & b_{1m} \\ c'_{12} & c'_{22} & \dots & c'_{n2} & b'_{12} & 1 & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c'_{1m} & c'_{2m} & \dots & c'_{nm} & b'_{1m} & b'_{2m} & \dots & 1 \end{array}\right]$$

with integral nonnegative coefficients. We say that C is *symmetrizable* by positive natural numbers $f_1, \dots, f_n, g_1, \dots, g_m$ if $a_{ij}f_j = f_i a'_{ij}$, $b_{ij}g_j = g_i b'_{ij}$, and $c_{ij}g_j = f_i c'_{ij}$ for all i, j. In this situation we construct the bilinear form

$$(4.6) \qquad \langle \cdot, - \rangle_C \colon \mathbb{Z}^{n+m} \times \mathbb{Z}^{n+m} \to \mathbb{Z}$$

given by the formula

$$\langle x, y\rangle_C = \sum_{i,j=1}^{n} a_{ij} f_j x_i y_j - \sum_{i=1}^{n}\sum_{s=1}^{m} c_{is} g_s x_i y_{n+s} + \sum_{s,t=1}^{m} b_{st} g_t x_{n+s} y_{n+t},$$

where $a_{ii} = 1 = b_{ss}$. Let $\chi_C(z) = \langle z, z\rangle_C$ be the quadratic form associated to $\langle \cdot, -\rangle_C$ and let

$$(-, \cdot)_C \colon \mathbb{Z}^{n+m} \times \mathbb{Z}^{n+m} \to \mathbb{Z}$$

be the associated symmetric bilinear form, i.e., $(z, w)_C = \frac{1}{2}(\langle z, w\rangle_C + \langle w, z\rangle_C)$.

Let $\xi_1, \dots, \xi_{n+m} \in \mathbb{Z}^{n+m} = \mathbb{Z}^n \oplus \mathbb{Z}^m$ be the standard basis (i.e., $\xi_i(j) = \delta_{ij}$ is the Kronecker delta). We define *reflections*

$$\delta_j \colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}, \qquad j = 1, \dots, n+m,$$

by the formulas

$$\delta_i(z) = z - \frac{2(z, \xi_i)_C}{f_i}\xi_i \quad \text{for } i \le n$$

and

$$\delta_{n+t}(z) = z - \frac{2(z, \xi_{n+t})_C}{g_t}\xi_{n+t} \quad \text{for } t = 1, \dots, m.$$

The composed map

$$(4.7) \qquad \delta = \delta_1 \cdots \delta_n \cdots \delta_{n+m}$$

is called the *Coxeter transformation* associated to C. Clearly, $\chi_C(\delta(z)) = \chi_C(z)$ for any vector $z \in \mathbb{Z}^{n+m}$.

Lemma 4.8. *Let* $z \in \mathbb{Z}^{n+m}$. *Then*

(a) $\delta(z)_j = \delta_j\delta_{j+1}\cdots\delta_{n+m}(z)_j$;
(b) $\langle \delta_{j+1}\cdots\delta_{n+m}(z), \xi_j\rangle_C = \langle z, \xi_j\rangle_C$.

Proof. (a) is straightforward.

(b) is by downwards induction on j. If $j = n+m$, there is nothing to show. Let $j < n+m$. We may assume that $n \leq j$; then

$$\langle \delta_{j+1}\cdots\delta_{n+m}(z), \xi_j\rangle_C = \left\langle \delta_{j+2}\cdots\delta_{n+m}(z) - \frac{2(z, \xi_{j+1})}{g_{j+1-n}}\xi_{j+1}, \xi_j \right\rangle_C.$$

Since $\langle \xi_{j+1}, \xi_j\rangle_C = 0$, the result follows by induction. □

The following result is similar to [16, 2.4].

Lemma 4.9. *For any two vectors* $v, w \in \mathbb{Z}^{n+m}$ *we have* $\langle w, \delta(v)\rangle_C = -\langle v, w\rangle_C$.

Proof. It is enough to show the formula for $w = \xi_j$, $j = 1, \dots, n+m$. Let $n < j \leq n+m$. By applying (4.8) we get

$$\begin{aligned}\langle \xi_j, \delta(v)\rangle_C &= g_{j-n}\delta(v)_j + \sum_{i=j-n+1}^{m} b_{(j-n)i}g_i\delta(v)_{n+i} \\ &= g_{j-n}\delta_{j+1}\cdots\delta_{n+m}(v)_j - 2(\delta_{j+1}\cdots\delta_{n+m}(v), \xi_j)_C \\ &\quad + \sum_{i=j-n+1}^{m} b_{(j-n)i}g_i\delta(v)_{n+i}.\end{aligned}$$

Since

$$\begin{aligned}2(\delta_{j+1}\cdots\delta_{n+m}(v), \xi_j)_C &= \langle v, \xi_j\rangle_C + \langle \xi_j, \delta_{j+1}\cdots\delta_{n+m}(v)\rangle_C \\ &= \langle v, \xi_j\rangle_C + \sum_{i=j-n}^{m} b_{(j-n)i}g_i\delta_{j+1}\cdots\delta_{n+m}(v)_{n+i},\end{aligned}$$

we get the desired result. The proof for $1 \leq j \leq n$ is similar. □

We say that R has a *schurian upper triangular form* if R has the shape

$$(4.10)\qquad R = \left[\begin{array}{ccc|ccc} A_1 & & {}_iA_j & {}_1M_1 & \dots & {}_1M_m \\ & \ddots & & \vdots & & \vdots \\ 0 & & A_n & {}_nM_1 & \dots & {}_nM_m \\ \hline & & & B_1 & & {}_sB_t \\ & 0 & & & \ddots & \\ & & & 0 & & B_m \end{array}\right],$$

where $A_1, \dots, A_n$ and $B_1, \dots, B_m$ are division rings, and ${}_iA_j$, ${}_iM_s$ and ${}_sB_t$ are bimodules. We assume that $A_i = e_iAe_i$, ${}_iA_j = e_iAe_j$, $B_s = \eta_sB\eta_s$, ${}_sB_t = \eta_sB\eta_t$, and ${}_iM_j = e_iR\eta_j$.

We associate to R the bipartite Cartan matrix $C = C(R)$ with $a_{ij} = \dim({}_iA_j)_{A_j}$, $a'_{ij} = \dim_{A_i}({}_iA_j)$, $c_{it} = \dim({}_iM_t)_{B_t}$, $c'_{it} = \dim_{A_i}({}_iM_t)$, $b_{st} = \dim({}_sB_t)_{B_t}$, and $b'_{st} = \dim_{B_s}({}_sB_t)$. Then $C(R)$ is symmetrizable by the numbers

$$f_j = \dim_k A_j \quad \text{and} \quad g_t = \dim_k B_t.$$

It is clear that

(4.11) $$\langle \cdot, - \rangle_R = \langle \cdot, - \rangle_{C(R)}.$$

From here on, we assume that R is an *indecomposable ring* (or connected in the sense of [9]). In this case, by well-known arguments of Auslander, R is of finite prinjective type if and only if $\Gamma(\text{prin}(R)_B^A)$ has a finite component.

Lemma 4.12. *Assume that R is of finite prinjective type and $\Gamma(\text{prin}(R)_B^A)$ has a preprojective component. Then:*

(a) *R has (up to isomorphism) a schurian upper triangular form* (4.10).
(b) *$C(R)$ has the property that $c_{is}c'_{is} \le 3$ for all i, s. If in addition R is an algebra over an algebraically closed field, then $c_{is}c'_{is} \le 1$.*

Proof. (a) Any cycle $\eta_{i_0}B\eta_{i_1} \ne 0$, $\eta_{i_1}B\eta_{i_2} \ne 0, \ldots, \eta_{i_l}B\eta_{i_0} \ne 0$ would produce a cycle $\text{Hom}_R(P_0, P_l) \ne 0$, $\text{Hom}_R(P_l, P_{l-1}) \ne 0, \ldots, \text{Hom}_R(P_1, P_0) \ne 0$ and thus is in $\Gamma(\text{prin}(R)_B^A)$. Here, $P_s = (0, \mathfrak{N}(\eta_{i_s}B), 0) \in \text{prin}(R)_B^A$.) If $x \in \eta_s B \eta_s$ (resp. $x \in e_j A e_j$) is noninvertible, then we get a noninvertible endomorphism of P_s (resp. of $(e_j A, 0, 0)$).

(b) Let $1 \le i \le n$ and $1 \le s \le m$. Let

$$R' = \begin{bmatrix} A_i & {}_iM_s \\ 0 & B_s \end{bmatrix}.$$

There is a fully-faithful functor $\varphi\colon \text{prin}(R')_{B_s}^{A_i} \to \text{prin}(R)_B^A$ such that

$$\text{res} \circ \varphi(X) \xrightarrow{\sim} X$$

(compare with [18, 19]). Therefore, R' is of finite prinjective type. The Tits form associated with $\text{prin}(R')_{B_s}^{A_i}$ is

$$f_i x^2 + g_s y^2 - c_{is} g_s xy.$$

By (4.2), this form is weakly positive; then the discriminant

$$c_{is}^2 g_s^2 - 4 f_i g_s = c_{is} c'_{is} f_i g_s - 4 f_i g_s$$

is negative. Thus, $c_{is}c'_{is} < 4$.

If R is a k-algebra with k algebraically closed, then $f_i = 1 = g_s$ for all i, s. Therefore, $c_{is} = c'_{is}$ and the condition above implies $c_{is}c'_{is} \le 1$. □

A large class of examples of rings R satisfying the assumptions of the above lemma are the *sp*-representation-finite right peak algebras studied in [14] and the piecewise peak algebras of finite prinjective type studied in [20]. In both cases there is a diagrammatic characterization of these algebras and sincere algebras are described.

Proposition 4.13. *Let R be a ring of the form* (0.1) *and assume that $\Gamma(\text{prin}(R)_B^A)$ has a preprojective component. Then R is of finite prinjective type if and only if χ_R is weakly positive. Moreover, in this case, if R is a k-algebra with k an algebraically closed field, then:*

(a) *$X \mapsto$ **cdn** X is a bijection between the indecomposable modules in* $\text{prin}(R)_B^A$ *and the positive roots of χ_R.*
(b) *If X and Y are indecomposables in* $\text{prin}(R)_B^A$ *and* **dim** $X =$ **dim** Y, *then $X \simeq Y$.*

Proof. The first claim and (a) follow from the well-known Drozd's arguments [8, 16].

(b) Consider the matrix

$$H = \left[\begin{array}{c|c} (\dim_k e_i A e_j)_{i,j} & (\dim_k e_i M \eta_s)_{i,s} \\ \hline 0 & (\dim_k \eta_s B \eta_t)_{s,t} \end{array}\right].$$

Then we have

$$\left((\dim_k X' e_i)_i, \left(\sum_{i=1}^{n} (\dim_k X' e_i)(\dim_k e_i M \eta_s) + \dim_k X''_{\eta_s} \right)_s \right) = (\mathbf{cdn}\ X)H.$$

By (4.12), H is invertible and the result follows from (a). □

Our aim is to use the Coxeter transformation δ associated with $C(R)$ to calculate the Auslander-Reiten translation in $\operatorname{prin}(R)^A_B$. We start by showing that in some cases δ preserves positiveness of vectors.

Lemma 4.14. *Let R be of finite prinjective type over an algebraically closed field and such that $\Gamma(\operatorname{prin}(R)^A_B)$ has a preprojective component. Let $v \in \mathbb{N}^{n+m}$ be such that $\chi_R(v) = 1$. Then:*

(a) $-1 \le 2(v, \xi_i)_R \le 1$ *for* $1 \le i \le n+m$ *with* $v_i \ne 0$.

(b) *If v is a sincere vector (that is, $v(i) > 0$ for every $1 \le i \le n+m$), then* $\delta(v) \in \mathbb{N}^{n+m}$.

Proof. (a) Since $\chi_R(\xi_i) = 1$ and $v - \xi_i$ is a vector with nonnegative coordinates, $0 < \chi_R(v - \xi_i) = 2 - 2(v, \xi_i)_R$ and $2(v, \xi_i)_R \le 1$. Similarly, $-1 \le 2(v, \xi_i)_R$.

(b) By downwards induction on j $(0 \le j \le n+m)$ we show that

$$\delta_{j+1} \cdots \delta_{n+m}(v) \in \mathbb{N}^{n+m}.$$

If $j = n+m$, there is nothing to show. Let $j < n+m$ and $1 \le i \le n+m$. If $i < j$, $\delta_j \cdots \delta_{n+m}(v)_i = v_i > 0$. If $i \ge j$, by (4.8)

$$\begin{aligned} \delta_j \cdots \delta_{n+m}(v)_i &= \delta_i \cdots \delta_{n+m}(v)_i \\ &= \delta_{i+1} \cdots \delta_{n+m}(v)_i - 2(\delta_{i+1} \cdots \delta_{n+m}(v), \xi_i)_R \\ &= v_i - 2(\delta_{i+1} \cdots \delta_{n+m}(v), \xi_i)_R. \end{aligned}$$

By the induction hypothesis, $\delta_{i+1} \cdots \delta_{n+m}(v) \in \mathbb{N}^{n+m}$. By (a),

$$2(\delta_{i+1} \cdots \delta_{n+m}(v), \xi_i)_R \le 1.$$

Thus, $\delta_j \cdots \delta_{n+m}(v)_i \ge 0$. □

Theorem 4.15. *Let R be an algebra over an algebraically closed field k. Assume that R is of finite prinjective type and that $\Gamma(\operatorname{prin}(R)^A_B)$ has a preprojective component. Let $V \in \operatorname{prin}(R)^A_B$ be an indecomposable module with $v = \mathbf{cdn}\ V$ such that $\delta(v) \in \mathbb{N}^{n+m}$. Then $\delta(v) = \mathbf{cdn}\ \Delta V$.*

Proof. Since $\delta(v)$ is a positive root of χ_R, by (4.13) there exists an indecomposable module $V' \in \operatorname{prin}(R)^A_B$ with $\delta(v) = \mathbf{cdn}\ V'$.

We claim that ΔV is the unique indecomposable in $\operatorname{prin}(R)^A_B$ satisfying:

(i) $\langle v, \mathbf{cdn}\ \Delta V\rangle_R < 0$.

(ii) If W is an indecomposable with $\langle v, \mathbf{cdn}\ W\rangle_R < 0$, then

$$\langle \mathbf{cdn}\ W, \mathbf{cdn}\ \Delta V\rangle_R > 0.$$

For, since $\Gamma(\text{prin}(R)^A_B)$ is a preprojective component, then $\text{Hom}_R(V, \Delta V) = 0$ and by (4.4),

$$\langle v, \mathbf{cdn}\,\Delta V\rangle_R = -\dim_k \text{Ext}^1_R(V, \Delta V) < 0.$$

The property defining Auslander-Reiten sequences implies (ii). Moreover, if Z is an indecomposable satisfying (i) and (ii), we should have $\langle \mathbf{cdn}\,Z, \mathbf{cdn}\,\Delta V\rangle_R > 0$ and $\langle \mathbf{cdn}\,\Delta V, \mathbf{cdn}\,Z\rangle_R > 0$. Hence, $\text{Hom}_R(Z, \Delta V) \neq 0 \neq \text{Hom}_R(\Delta V, Z)$, and therefore $Z \simeq \Delta V$.

Now, note that by (4.9) the vector $\delta(v) = \mathbf{cdn}\,V'$ satisfies (i) and (ii). It follows that $V' \simeq \Delta V$ and $\delta(v) = \mathbf{cdn}\,\Delta V$. □

Corollary 4.16. *Let R and V be as in* (4.15). *If $v = \mathbf{cdn}\,V$ is sincere, then* $\delta(v) = \mathbf{cdn}\,\Delta V$.

Proof. The proof follows from (4.14) and (4.15). □

Remark 4.17. Let R be a basic, indecomposable triangular k-algebra of the form (4.10) and assume that k is algebraically closed.

(a) It is possible to give an algorithmic construction of the modules on a preprojective component of $\Gamma(\text{prin}(R)^A_B)$. This generalization of (4.15) makes use of a nonlinear map δ' instead of δ. The construction follows that given in [12] for the case of algebras.

(b) We say that R is *sincere* if its Tits form χ_R has a sincere root. The *separation-criterion* for R can be defined as in the case of algebras. If R satisfies the separation-criterion, then $\Gamma(\text{prin}(R)^A_B)$ has a preprojective component. There is an algorithmic construction of all the k-algebras R which are sincere, of finite-prinjective type, and which satisfy the separation-criterion.

We will come back to these and other algorithmic problems in a forthcoming publication.

Note added in proof

In the proof of Theorem 3.4 the following situation remains to be considered: Suppose that $\Theta_B(X)$ is projective in $\text{adj}(R)^A_B$. Since X is not projective in $\text{mod}_{ic}(R)$, then X is not of the form $\widehat{e_iR}$ $(i = 1, \dots, n)$ and therefore we conclude from Lemma 3.3 that there exists $j \in \{1, \dots, m\}$ such that $\Theta_B(X) \simeq \Theta^A\nabla_-(\widehat{\eta_j R^\nabla})$, where R^∇ is defined in (3.6) and

$$\nabla_-\colon \text{mod}_{ic}(R^\nabla)_A \to \text{mod}^{pg}(R)^A$$

is an equivalence defined below (3.5). It follows from the definition of ∇_- that the inclusion $\text{rad}\,\widehat{\eta_j R^\nabla} \hookrightarrow \widehat{\eta_j R^\nabla}$ induces a commutative diagram

$$\begin{array}{ccccccccc} e_0\colon 0 & \to & \text{top}\,\eta_j R & \to & E_0 & \xrightarrow{w} & \Theta_B(X) & \to & 0 \\ & & \cap\!\!\downarrow & & \downarrow & & \downarrow \eta_X & & \\ \tilde{e}_0\colon 0 & \to & {}^0I_j & \to & L_0 & \xrightarrow{w'} & X \simeq \widehat{\Theta_B(X)} & \to & 0 \end{array}$$

with exact rows, where $E_0 = \Theta^A\nabla_-(\widehat{\text{rad}\,\eta_j R^\nabla})$ and L_0 is such that $\Theta_B(L_0) \simeq E_0$. Since ∇_- is an equivalence, w is a sink map in $\text{adj}(R)^A_B$ and one can easily prove as above that $\hat{e}_0$ is an Auslander-Reiten sequence in $\text{mod}_{ic}(R)_B$.

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